Kinematics-Chapter-4.nb

Written in *Mathematica* 12.2.0.0 Vince Cronin
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4.1 Working with the rotation of a spherical Earth

4.1.1 Rotation of a point on the Equator around a globe's spin axis.

We considered a simple coordinate transformation in two dimensions in chapter 3, where two Cartesian coordinate systems share a common origin. We extended that solution to three dimensions by specifying that both coordinate systems shared the same *Z* axis. Now, we will extend this knowledge to consider how to determine the location of a point defined in a coordinate system that is rotating relative to another coordinate system that shares a common origin and coordinate axis.

Imagine a room where there is a globe model of Earth mounted on a vertical rod that passes through the globe from its South Pole through the center to the globe's North Pole, at the top of the globe. You are seated immobile in the room, observing the globe, so the room provides a frame of reference for any future displacement of the globe around its axis. At the beginning of the experiment, you define a Cartesian coordinate system fixed to the room (the *room CS*) whose origin is at the center of the globe. The *Z* axis of the room CS extends up along the rod through the globe North Pole, the *X* axis extends through globe latitude 0° longitude 0°, the *Y* axis extends through globe latitude 0° longitude 90°E, and the globe radius is taken to be 1 unit.

At that same initial time, you also define a second Cartesian coordinate system that is identical to the first, and specify that the second coordinate system is fixed to the globe (the *globe CS*). In certain engineering applications, this is called an Earth-Centered Earth-Fixed coordinate system. If the globe rotates away from its initial position (as observed in the room CS), the globe CS rotates with it. The globe is a rigid sphere, by which we simply mean that it retains its spherical shape and size throughout the experiment. No point on the globe's surface moves relative to any other point on its surface during the experiment.

We label a point *P* located on the globe surface at latitude 0° longitude 0°, and label a second point *N* at the North Pole at the top of the globe. The unit location vectors to points *P* and *N* are, respectively,

$\hat{p} = \{1, 0, 0\}$ and

$\hat{n} = \{0, 0, 1\}$.

These vector coordinates are the same in both the room CS and the globe CS at the beginning of our experiment. The vector coordinates remain the same in the globe CS throughout the experiment, but they will change in the room CS if the globe rotates relative to the room.

Later, the globe has rotated in a positive (right-handed or anti-clockwise) direction by 30° around the spin axis, as observed from the room CS (Fig. 4.1A). That is, the globe CS and all points on the globe have rotated around the *Z* axis that is common to both the room CS and the globe CS, and both coordinate systems still share the same origin at the center of the globe. What are the Cartesian coordinates of point *P* in the room CS after the 30° rotation?

We can solve this problem by drafting accurate maps of the situation and making some trigonometric calculations, as we did in the similar scenario presented in section 3.2 (Fig. 4.1B). Using this longhand method, we find that the coordinates of point *P* in the room CS after a 30° rotation are

$\hat{p}_{\text{room}} = \{\cos(30°), \sin(30°), 0\} \simeq \{0.866025, 0.5, 0\}$

while the coordinates of point P in the globe CS are unchanged

$\hat{p}_{\text{globe}} = \{1, 0, 0\}$.

4.1.2 The transformation matrix for a rotation around a common Z axis

We introduced the concept of using a matrix of direction cosines to transform between two 3D Cartesian coordinate systems in section 3.4.2. Now, we will develop a matrix to help us with a problem where we have two Cartesian coordinate systems with a common origin and a common Z axis: one functions as our reference frame, and the other is fixed to a rigid body that rotates around the common Z axis.

We will label the basis vectors associated with the reference frame (the room CS in section 4.1.1) as $\hat{i}$, $\hat{j}$, $\hat{k}$, where $\hat{i}$ is the unit vector along the X_{rf} axis, $\hat{j}$ is the unit vector along the Y_{rf} axis, and $\hat{k}$ is the unit vector along the Z_{rf} axis. The basis vectors of the coordinate system fixed to the rigid body (the globe CS in section 4.1.1) are labeled $\hat{i}'$, $\hat{j}'$, $\hat{k}'$, where $\hat{i}'$ is the unit vector along the X_{rb} axis, $\hat{j}'$ is the unit vector along the Y_{rb} axis, and $\hat{k}'$ is the unit vector along the Z_{rb} axis. These two Cartesian coordinate systems share a common origin.

The two coordinate systems are fully aligned at the initial time, so $\hat{i} = \hat{i}'$, $\hat{j} = \hat{j}'$, and $\hat{k} = \hat{k}'$. At the later time, the angular relationships among the basis vectors of the two coordinate systems are as follows:

	$\hat{i}'$	$\hat{j}'$	$\hat{k}'$
$\hat{i}$	30 °	120 °	90 °
$\hat{j}$	60 °	30 °	90 °
$\hat{k}$	90 °	90 °	0

The components of the transformation matrix are a set of direction cosines — cosines of the angle between specified pairs of basis vectors

$$\begin{pmatrix} \text{cosine of angle between } \hat{i}' \text{ and } \hat{i} & \text{cosine of angle between } \hat{j}' \text{ and } \hat{i} & \text{cosine of angle between } \hat{k}' \text{ and } \hat{i} \\ \text{cosine of angle between } \hat{i}' \text{ and } \hat{j} & \text{cosine of angle between } \hat{j}' \text{ and } \hat{j} & \text{cosine of angle between } \hat{k}' \text{ and } \hat{j} \\ \text{cosine of angle between } \hat{i}' \text{ and } \hat{k} & \text{cosine of angle between } \hat{j}' \text{ and } \hat{k} & \text{cosine of angle between } \hat{k}' \text{ and } \hat{k} \end{pmatrix}$$

or, in this instance,

$$\begin{pmatrix} \cos(30°) & \cos(120°) & \cos(90°) \\ \cos(60°) & \cos(30°) & \cos(90°) \\ \cos(90°) & \cos(90°) & \cos(0°) \end{pmatrix}$$

We already know the angles between these pairs of basis vectors, and so we know the cosines of those angles. In particular, we know that cos(90°) = 0 and cos(0) = 1, so we can just insert those values into the matrix and leave the other elements expressed as cosines:

$$\begin{pmatrix} \cos(30°) & \cos(120°) & 0 \\ \cos(60°) & \cos(30°) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For this type of problem in which we are rotating a rigid body around one of the coordinate axes of the frame of reference, we would like to use the rotation angle θ as the only input value. For any angle θ, $\cos(90°+\theta) = -\sin(\theta)$ and $\cos(90-\theta) = \sin(\theta)$. Hence, the transformation matrix J for a rotation of θ around the Z axis (that is, for a rotation of θ around the $\hat{k}' = \hat{k}$ basis vectors) is

$$J = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, let us see if this transformation matrix provides a path to the solution of the problem posed in section 4.1.2 in which unit location vector p_{globe} is rotated in a positive (anticlockwise) direction by 30° around the common Z axis of the room CS and the globe CS. Our input data are

$$\hat{p}_{\text{globe}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \theta = 30°, \text{and } J = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So he location of point P in the room CS after a rotation of 30° is

$$\hat{p}_{\text{room}} = \begin{pmatrix} \cos(30°) \\ \sin(30°) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(30°) & -\sin(30°) & 0 \\ \sin(30°) & \cos(30°) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\hat{p}_{\text{room}} \simeq \{0.866025, 0.5, 0\}$.

These are the same coordinates we obtained manually in section 4.1.1. But is it the correct answer? At the initial time, point P was located on the Equator at latitude 0° and longitude 0°, and had a unit location vector $\hat{p}_{\text{globe}} = \{1, 0, 0\}$. The point that was +30° away on the Equator was located at latitude 0° and longitude 30°. Converting geographic coordinates to Cartesian geographic coordinates (see section 2.2.3), the point on the Equator that was +30° away at the initial time had a location vector of {cos(0°)cos(30°), cos(0°)sin(30°), sin(0°)} or approximately{0.866025, 0.5, 0}. Those are the same vector coordinates for p_{room} that we computed in two different ways.

In *Mathematica*, the corresponding code (written in list form rather than in matrix form) looks like the following.

```
vectorPglobe={1,0,0};
theta=30;
matrixJ={{Cos[theta Degree],-Sin[theta Degree],0},{Sin[theta Degree],Cos[-
theta Degree],0},{0,0,1}};
vectorProom=matrixJ.vectorPglobe;
N[vectorProom]
```

4.1.3 Rotation of an arbitrary set of points around the globe's spin axis

Would the transformation matrix *J* that was defined in section 4.1.2 for a rotation of point *P* around the *Z* coordinate axis work for other starting points? We will pick two other points on the globe, determine their distance from one another and from point *P* at the initial time, use matrix *J* to rotate them 30° around the *Z* axis, and determine if the distances between the points are the same after rotation as they were initially. The three points will be called *P*, *S*, and *U*, with unit location vectors as defined in the globe CS as follows:

$$\hat{p}_{\text{globe}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \hat{s}_{\text{globe}} = \begin{pmatrix} 0.627507 \\ 0.439385 \\ 0.642788 \end{pmatrix}, \text{and } \hat{u}_{\text{globe}} = \begin{pmatrix} 0.875426 \\ 0.234570 \\ -0.422618 \end{pmatrix}$$

At the initial time, the angular distances between these points are as follows (see section 2.2.9):

$\hat{p}_{\text{globe}}$ to $\hat{s}_{\text{globe}} \simeq 51.1°$ $\hat{p}_{\text{globe}}$ to $\hat{u}_{\text{globe}} \simeq 28.9°$ $\hat{u}_{\text{globe}}$ to $\hat{s}_{\text{globe}} \simeq 67.6°$

A rotation of the globe by +30° as observed in the room CS brings these points to the following coordinates

$$\hat{p}_{\text{room}} = \begin{pmatrix} 0.866025 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(30°) & -\sin(30°) & 0 \\ \sin(30°) & \cos(30°) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{s}_{\text{room}} = \begin{pmatrix} 0.323744 \\ 0.694272 \\ 0.642788 \end{pmatrix} = \begin{pmatrix} \cos(30°) & -\sin(30°) & 0 \\ \sin(30°) & \cos(30°) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.627507 \\ 0.439385 \\ 0.642788 \end{pmatrix}$$

$$\hat{u}_{\text{room}} = \begin{pmatrix} 0.640856 \\ 0.640856 \\ -0.422618 \end{pmatrix} = \begin{pmatrix} \cos(30°) & -\sin(30°) & 0 \\ \sin(30°) & \cos(30°) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.875426 \\ 0.234570 \\ -0.422618 \end{pmatrix}$$

The angular distance between these points after rotation are as follows:

$\hat{p}_{\text{room}}$ to $\hat{s}_{\text{room}} \simeq 51.1°$ $\hat{p}_{\text{room}}$ to $\hat{u}_{\text{room}} \simeq 28.9°$ $\hat{u}_{\text{room}}$ to $\hat{s}_{\text{room}} \simeq 67.6°$

The distance between these three points is unchanged after rotation, as we would expect during a rigid-body rotation around an axis.

In *Mathematica*, the corresponding code (written in list form rather than in matrix form) looks like the following.

```
vectorPglobe={1,0,0};
vectorSglobe={0.627507,0.439385,0.642788};
vectorUglobe={0.875426,0.234570,-0.422618};
theta=30;

matrixJ={{Cos[theta Degree],-Sin[theta Degree],0},{Sin[theta Degree],Cos[-
theta Degree],0},{0,0,1}};

vectorProom=matrixJ.vectorPglobe;
N[vectorProom]
vectorSroom=matrixJ.vectorSglobe;
```

```
N[vectorSroom]
vectorUroom=matrixJ.vectorUglobe;
N[vectorUroom]

angleProom2Sroom=VectorAngle[vectorProom,vectorSroom](180/π);
N[angleProom2Sroom]
angleProom2Uroom=VectorAngle[vectorProom,vectorUroom](180/π);
N[angleProom2Uroom]
angleUroom2Sroom=VectorAngle[vectorUroom,vectorSroom](180/π);
N[angleUroom2Sroom]
```

Exercise 4.1 (HW-06) Consider four points (A, B, C, and D) on the globe whose initial locations in the room CS and globe CS are as follows: $\text{latitude}_A = 16°$, $\text{longitude}_A = 38°$; $\text{latitude}_B = 34°$, $\text{longitude}_B = -26°$; $\text{latitude}_C = -44°$, $\text{longitude}_C = -37°$; and $\text{latitude}_D = -19°$, $\text{longitude}_D = 43°$. Using the template provided, complete a *Mathematica* notebook that computes the location vectors for this set of points in the room CS (a) after a rotation of 47° around the North Pole from the initial position, and (b) after a rotation of -56° around the North Pole from the initial position.

4.1.4 Rotation matrices

We started with the development of a transformation matrix to help us navigate between Cartesian coordinate systems that share a common origin. We focused on instances when the two coordinate systems also share a common *Z* axis, and developed a specialized transformation matrix that helps us to determine the coordinates of a vector that has been rotated through an angle θ around the *Z* axis. We will refer to this sort of matrix generally as a rotation matrix, and the rotation matrix around a common *Z* axis will be called a Z-rotation matrix, where

$$\text{Z-rotation matrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using a similar process, we can define an X-rotation matrix

$$\text{X-rotation matrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

and a Y-rotation matrix

$$\text{Y-rotation matrix} = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}.$$

4.2 Angular velocity

4.2.1 Introduction to angular velocity in a geocentric coordinate system

In order to discuss velocity, we need to specify a coordinate system. Velocity can be described as a vector, so it has a magnitude and a direction. The vector from an initial position to a subsequent position is called the displacement vector, and its magnitude is the distance between the two positions. The quantitative description of direction and distance requires a spatial coordinate system. The associated velocity vector has the same direction as the displacement vector but its magnitude is the time it took to move from the initial position to the final position. Time might be considered the fourth dimension of our analytical world. The magnitude of a velocity vector is called its speed.

The instantaneous motion of a lithospheric plate and the average motion over a given time interval are described quantitatively as a rotation rate around a geocentric axis. In the plate kinematics literature, that rate is expressed typically in degrees per million years. So if we have an angular velocity and divide it by a time interval, the result will be an angle. We will let the scalar $[\![\omega]\!]$ denote the angular speed in degrees per million years, and the scalar t represents time. We might consider ω to be an angular velocity vector with a geocentric origin, oriented parallel to a rotation axis, directed toward the point on Earth's surface (the pole of rotation) around which a part of that surface is rotating in a positive (anticlockwise) sense, and whose magnitude $[\![\omega]\!]$ is the angular speed.

4.2.2 Using angular velocity and elapsed time to specify the angle in the rotation matrix

We can recast the Z-rotation matrix in terms of angular speeds as follows:

$$\text{Z-rotation matrix} = \begin{pmatrix} \cos([\![\omega]\!]\,t) & -\sin([\![\omega]\!]\,t) & 0 \\ \sin([\![\omega]\!]\,t) & \cos([\![\omega]\!]\,t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Earth rotates around its axis once in 24 hours, as observed in a geocentric Cartesian coordinate system within which the centers of Earth and the Sun are fixed (that is, the centers of Earth and the Sun are not moving in this reference frame). We'll call this the Earth-Sun coordinate system. A second coordinate system is the Cartesian geographic coordinate system that is fixed to the solid Earth (Earth CS). The Earth-Sun CS and the Earth CS are fully aligned at the beginning of the experiment when $t = 0$. The angular velocity vector is directed at the North Pole, and its magnitude $[\![\omega]\!]$ is 360°/24 hrs, or 15°/hr. We can compute where any point P on Earth will be in the Earth-Sun CS at any time in the future ($+t$) or past ($-t$) given its initial position (at t=0) in the Earth-Sun CS. If we define the location of point P at time 0 (t_0) with the location vector $\hat{p}_{t_0}$ and its location at some other time (t_n) as $\hat{p}_{t_n}$ where

$$\hat{p}_{t_0} = \begin{pmatrix} p_{1\,t_0} \\ p_{2\,t_0} \\ p_{3\,t_0} \end{pmatrix}, \text{ then}$$

$$\hat{p}_{t_n} = \begin{pmatrix} p_{1\,t_n} \\ p_{2\,t_n} \\ p_{3\,t_n} \end{pmatrix} = \begin{pmatrix} \cos(15\,°/\text{hr}\;\; t_\text{n}) & -\sin(15\,°/\text{hr}\;\; t_\text{n}) & 0 \\ \sin(15\,°/\text{hr}\;\; t_\text{n}) & \cos(15\,°/\text{hr}\;\; t_\text{n}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{1\,t_0} \\ p_{2\,t_0} \\ p_{3\,t_0} \end{pmatrix}$$

Example. Let $\hat{p}_{t_0} = \{1, 0, 0\}$, $\omega = 15°/\text{hr}$ around the North Pole (that is, around the Z axis of the Earth CS and Earth-Sun CS), and $t_\text{n} = 2$ hr. What is are the coordinates of location vector $\hat{p}_{t_n}$ in the Earth-Sun CS after 2 hours of rotation?

$$\hat{p}_{t_n} = \begin{pmatrix} 0.866025 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(15° \times 2) & -\sin(15° \times 2) & 0 \\ \sin(15° \times 2) & \cos(15° \times 2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

where the × symbol is used to indicate scalar multiplication in this instance.

In *Mathematica*, the corresponding code (written in list form rather than in matrix form) looks like the following, where the * symbol is used to indicate scalar multiplication.

```
vectorPt0={1,0,0};
omega=15;
tn=2;

matrixJ={{Cos[(omega*tn) Degree],-Sin[(omega*tn)
Degree],0},{Sin[(omega*tn) Degree],Cos[(omega*tn) Degree],0},{0,0,1}};

vectorPtn=matrixJ.vectorPt0;
N[vectorPtn]
```

Exercise 4.2 (HW-07) Consider a point A whose current location in the Earth CS and Earth-Sun CS is as follows: $\text{latitude}_A = 16°$, $\text{longitude}_A = 38°$. Given Earth's angular velocity of 15°/hr around the North Pole, complete a *Mathematica* notebook that computes the location vector for point A in the Earth-Sun CS (a) five hours in the future, and (b) three hours in the past.

4.3 Tangential velocity

4.3.1 Introduction to tangential velocity on a spherical Earth

We can define a tangent plane for every point on the surface of a spherical Earth. The tangent plane at any given point on Earth's surface is perpendicular to (that is, normal to) the unit location vector to that point. The tangent plane at a given point on the surface of a perfect sphere is the only point of contact between that plane and the sphere.

A point on the Equator of our idealized spherical Earth travels one full circumference every 24 hours, as viewed in the Earth-Sun coordinate system described in section 4.2.2. We take the mean Earth radius as 6371.01 km, so our idealized Earth has a circumference of $2\,\pi\,6371.01$ km or 40,030.24 km. Hence a point on the Equator moves at a constant rate of (40,030.24 km)/(24 hr) = 1667.93 km/hr perpendicular to Earth's spin axis and toward the east as observed in the Earth-Sun CS. The vector that represents that instantaneous velocity at a point on the Equator has a magnitude of nearly 1668 km/hr, is directed east, and is perpendicular to the location vector to that point and to Earth's spin axis. That instantaneous velocity vector is tangential to Earth's surface at the point where the velocity is defined.

4.3.2 Tangential velocities at different latitudes on Earth

If the tangential velocity on the Equator of our idealized spherical Earth is nearly 1668 km/hr toward due east in the Earth-Sun coordinate system, what is the tangential velocity at the North Pole? The North Pole is along the spin axis of Earth, which defines the *Z* axis of the Cartesian geographic coordinate system in which the centers of both Earth and the Sun are fixed. While the North Pole certainly rotates in the Earth-Sun CS, its location vector is unchanged so it experiences no displacement in that coordinate system. The tangential velocity of the North Pole in the Earth-Sun CS is zero.

What about the tangential velocities between the Equator and the North and South Poles? The tangential speed varies with the cosine of the latitude, between the maximum value at the Equator to zero at the poles. Hence, the tangential velocity for a point at latitude θ, moving east relative to the Earth-Sun CS, is

tangential velocity (km/hr) = $\cos(\theta)*((2*\pi*6371.01)/24)$

where the * symbol is used to indicate scalar multiplication and the term "$((2*\pi*6371.01)/24)$" is the circumference of Earth in km divided by 24 hours.

4.3.3 What is the tangential velocity at a given angular distance from the pole of rotation?

Let's ignore the existence of the geographic coordinate system of latitudes and longitudes, and solve the problem we encountered in section 4.3.2 referring only to the angle between the pole of positive rotation and the point whose tangential velocity we seek to find. In the Earth-Sun CS, the location vector to the North Pole has coordinates $\{0, 0, 1\}$. Let's use ϕ to signify the angle between the location vector to the North Pole and the location vector to some other point on Earth's surface, so ϕ varies from 0° at the North Pole to 180° at the South Pole. As observed in the Earth-Sun CS, a point that is an angular distance of ϕ from the North Pole moves east and its tangential speed in km/hr is

tangential speed = $\sin(\phi)*((2*\pi*6371.01)/24)$

where the * symbol is used to indicate scalar multiplication, and the term "$((2*\pi*6371.01)/24)$" is the circumference of Earth in kilometers divided by 24 hours, yielding kilometers per hour.

Now let's consider a slightly different situation, in which we are given an angular velocity. Earth rotates in the Earth-Sun CS at an angular speed $[\![\omega]\!]$ = 15°/hr around an axial vector pointing toward the North Pole (recall section 4.2.1). As observed in the Earth-Sun CS, a point that is an angular distance of ϕ from the North Pole moves east and its tangential speed in km/hr is

tangential speed = $\sin(\phi)\ ([\![\omega]\!]\ ((2\ \pi\ 6371.01)/360))$

where the term "$((2*\pi*6371.01)/360)$" is the circumference of Earth in kilometers divided by 360 degrees, yielding kilometers per degree of circumference.

References

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