

Vectors and Vector Arithmetic

Definitions. A simple (but incomplete) definition of a *vector* is that it is a mathematical entity that has a specific magnitude and direction. For example, *velocity* is a vector quantity because it has *magnitude* (that is, displacement defined for a specified time interval) and *direction*. The magnitude of a velocity vector is called the *speed*, and is represented by a single number. Hence, speed is a *scalar* quantity, and is not a vector quantity because speed contains no information about the direction of movement.

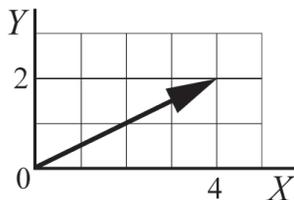
Scalars and vectors are both members of the mathematical family of entities called *tensors*. A scalar is a 0th rank tensor, and a vector is a 1st rank tensor. Tuck these factoids away for future use when we start working with tensors.

Vectors are commonly visualized as directed line segments or arrows. The length of the arrow is the magnitude, and the direction that the arrow is pointing is the vector's direction. The vector's head is the pointed end of the arrow, and the vector's origin is the other end that corresponds to where the nock and feathers are on the weapon sort of arrow.



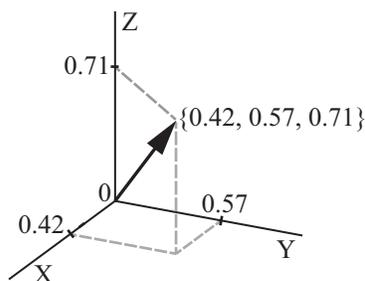
In order to know the direction that a vector is pointing, you need to specify a frame of reference. The direction of a vector on a horizontal plane might be specified by the azimuth or compass direction in which it is pointing. We will generally specify the direction of a vector using a right orthogonal Cartesian coordinate system. The *right* in that coordinate system means that we use the right-hand rule to define the positions of the positive X , Y and Z axes relative to each other. Starting at the positive X axis, the positive Y axis is found by a positive (that is, right-handed or counter-clockwise) rotation around the positive Z axis. Starting at the positive Y axis, the Z axis is found by a positive rotation around the X axis. And starting at the Z axis, the positive X axis is found by a positive rotation around the positive Y axis. "Orthogonal" means the axes are at right angles to one another, and are scaled the same so that 1 unit along one axis is the same length as 1 unit along the other axes. And these coordinate systems are "Cartesian" because they were first described and used by Rene Descartes.

In 2 dimensions, a vector might be represented graphically as in the following illustration



The origin of the vector in this example is drawn at the origin of the coordinate system, and the head of the vector is at a point whose $\{x, y\}$ coordinates are $\{4, 2\}$. We would describe the coordinates of the vector above as $\{4, 2\}$.

Many times, we will need to specify the direction of a 3-dimensional vector, so we need to refer to a 3D reference frame.



If we take any vector and imagine its origin located at the origin of a 3D Cartesian coordinate system, everything we need to know about the vector can be derived from the coordinates of the point at the head of the vector. In the illustration above in which the vector is represented by an arrow, the X coordinate of the head of the vector is 0.42, the Y coordinate is 0.57, and the Z coordinate is 0.71. In other words, the coordinates of the head of the vector are $\{0.42, 0.57, 0.71\}$. If we call that vector a , then

$$a = \{0.42, 0.57, 0.71\}.$$

We say that "the coordinates of vector a are $\{0.42, 0.57, 0.71\}$," meaning that the origin of vector a is located at $\{0, 0, 0\}$ and the head of vector a is located at $\{0.42, 0.57, 0.71\}$. We could move vector a so that its origin was in a different location, say $\{x, y, z\}$, and then its head would be located at $\{0.42+x, 0.57+y, 0.71+z\}$ for any values of x, y or z , respectively. After any such translation, vector a would still have the same direction and magnitude.

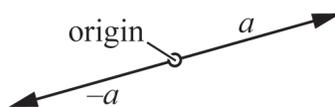
During a particular analysis, we might want to use symbols alone rather than inserting values for the vector coordinates. Two common ways of representing the vector coordinates symbolically are as follows:

$$a = \{a_x, a_y, a_z\},$$

which links each coordinate explicitly to coordinate axes that are labeled with the familiar X - Y - Z notation, or

$$a = \{a_1, a_2, a_3\}.$$

For a vector $a = \{a_1, a_2, a_3\}$, the opposite or negative of vector a (that is, $-a$) is a vector that has the same magnitude as a but is pointed in the opposite direction from a .



The resulting coordinates for the negative of vector a are $-a = \{-a_1, -a_2, -a_3\}$.

Example. What are the coordinates of vector $-a$ if the coordinates of vector a are $\{3, -2, 5\}$?

Answer: The coordinates of vector $-a$ are $\{-3, 2, -5\}$.

The magnitude (length) of a vector $a = \{a_1, a_2, a_3\}$ is commonly signified by $\|a\|$, which you might confuse with the symbol for absolute value.

$$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Example. If the coordinates of vector a are $\{3, -2, 5\}$, what is the magnitude of a ?

Answer: $\|a\| = \sqrt{3^2 + (-2)^2 + 5^2} \approx 6.16$

A *unit vector* is a vector whose magnitude is exactly 1. The unit vector of a vector a is signified by \hat{a} , and has coordinates

$$\hat{a} = \left\{ \frac{a_1}{\|a\|}, \frac{a_2}{\|a\|}, \frac{a_3}{\|a\|} \right\}$$

Example. If the coordinates of vector a are $\{3, -2, 5\}$, what are the coordinates of unit vector \hat{a} ?

Answer: $\hat{a} = \left\{ \frac{3}{6.16}, \frac{-2}{6.16}, \frac{5}{6.16} \right\} \approx \{0.49, -0.32, 0.81\}$

Vector Arithmetic

In the following subsections, we will use the following vectors:

$$a = \{a_1, a_2, a_3\}$$

$$b = \{b_1, b_2, b_3\}$$

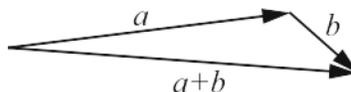
$$c = \{c_1, c_2, c_3\}$$

Vector addition. We can add a scalar s to a vector a , and the resulting coordinates are $\{(s+a_1), (s+a_2), (s+a_3)\}$.

To add two vectors together, you add the first component of the two vectors together to find the first component of the vector sum, then do the same for the second and third components.

$$a + b = \{(a_1 + b_1), (a_2 + b_2), (a_3 + b_3)\}$$

Graphically, vector addition looks like this:



If you have a set of n vectors (where n is an integer that is greater than 1), the result of adding all n vectors together is a vector whose direction is the mean (average) direction of that set of vectors. For example, the mean vector of the set of $n = 3$ vectors that includes vectors a , b and c is given by

$$a + b + c = \{(a_1 + b_1 + c_1), (a_2 + b_2 + c_2), (a_3 + b_3 + c_3)\}$$

Example. Given vectors $a = \{3, -3, 5\}$, $b = \{1, 4, 2\}$ and $c = \{-2, 5, -4\}$, what is the mean vector d of this set of three vectors?

Answer: $d = a + b + c$, so

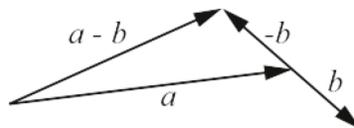
$$d = \{(3+1+(-2)), ((-3)+4+5), (5+2+(-4))\} = \{2, 6, 3\}$$

Vector subtraction. We can subtract a scalar s from a vector a , and the resulting coordinates are $\{(a_1 - s), (a_2 - s), (a_3 - s)\}$.

To subtract one vector from another, you subtract the first component of the first vector from the first component of the second vector to find the first component of the vector difference, then do the same for the second and third components.

$$a - b = \{(a_1 - b_1), (a_2 - b_2), (a_3 - b_3)\}$$

This is equivalent to adding vector a to the opposite of vector b (that is, adding a to $-b$). Graphically, vector subtraction looks like this:



Example. Given vectors $a = \{3, -3, 5\}$ and $b = \{1, 2, 4\}$, what is the difference $a - b$?

Answer: $a - b = \{(3-1), ((-3)-2), (5-4)\} = \{2, -5, 1\}$

Vector multiplication. Vectors can be "multiplied" in several senses. Vector $a = \{a_1, a_2, a_3\}$ can be multiplied by a scalar s as follows, using the symbol "*" to represent multiplication.

$$s*a = \{(s*a_1), (s*a_2), (s*a_3)\}.$$

Example. Given vector $a = \{3, -3, 5\}$ and scalar $s = 4$. What is the product of $s*a$?

Answer: $s*a = \{(4*3), (4*(-3)), (4*5)\} = \{12, -12, 20\}$

The effect of multiplying a vector by a positive scalar is to change the magnitude (length) of the vector without changing its direction; multiplying by a negative scalar reverses the vector's direction.

A vector **dot product** yields a single number. Because the dot product yields a single number, it is also called a scalar product. The dot product of two different vectors a and b is usually represented by the symbol $a \cdot b$. One way to compute the dot product $a \cdot b$ is

$$a \cdot b = (a_1b_1) + (a_2b_2) + (a_3b_3).$$

Example. Given vectors $a = \{3, -3, 5\}$ and $b = \{1, 2, 4\}$, what is the result of the vector dot product $a \cdot b$?

Answer: We will use the symbol "*" to indicate the multiplication of one number by another number.

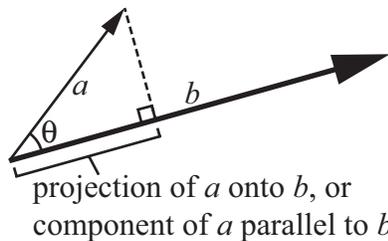
$$\begin{aligned} a \cdot b &= (3*1) + ((-3)*2) + (5*4) \\ &= 3 + (-6) + 20 = 17 \end{aligned}$$

Another way to compute the same dot product is given by the following.

$$a \cdot b = \|a\| \|b\| \cos \theta,$$

where $\|a\|$ is the length of vector a , $\|b\|$ is the length of vector b , and θ is the angle between vectors a and b . Angle θ is always less than or equal to 180° (π radians). Again, the result of a dot product is a scalar, because all three of the entities that are multiplied together ($\|a\|$, $\|b\|$, and $\cos \theta$) are scalars.

A dot product $a \cdot b$ can be thought of as the product of the projection of vector a onto b ($\|a\| \cos \theta$) times the length of b ($\|b\|$), which is also equal to the projection of b onto a ($\|b\| \cos \theta$) times the length of a ($\|a\|$). "The projection of vector a onto b " might also be described as the component of vector a that is parallel to vector b .



The dot product $a \cdot b$ is equal to the length of vector b times the length of the projection of vector a onto vector b

The angle θ between two vectors a and b is given by

$$\theta = \cos^{-1} \left[\frac{a \cdot b}{\|a\| \|b\|} \right]$$

Example. Given vectors $a = \{3, -3, 5\}$ and $b = \{1, 2, 4\}$, what is the angle θ between the two vectors?

Answer: $\|a\| = \sqrt{3^2 + (-3)^2 + 5^2} \approx 6.56$

$$\|b\| = \sqrt{1^2 + 2^2 + 4^2} \approx 4.58$$

$$a \cdot b = \{(3*1) + ((-3)*2) + (5*4)\} = 17$$

$$\theta = \cos^{-1} \left(\frac{17}{6.56 * 4.58} \right) \approx 0.97 \text{ radian}$$

$$\theta = 0.97 \left(\frac{180}{\pi} \right) \approx 55.55^\circ$$

The vector cross product, $a \times b$, results in a third vector whose direction is perpendicular to the plane defined by vectors a and b .

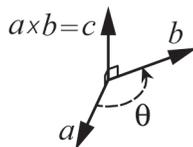
$$a \times b = \{(a_2 b_3) - (a_3 b_2), (a_3 b_1) - (a_1 b_3), (a_1 b_2) - (a_2 b_1)\}, \text{ and}$$

$$(a \times b) = -(b \times a)$$

The magnitude or length of the resulting vector is

$$\|a \times b\| = \|a\| \|b\| \sin \theta$$

where $\|a\|$ is the length of vector a , $\|b\|$ is the length of vector b , and θ is the angle between vectors a and b ; $\theta \leq 180^\circ$.



Example. Given vectors $a = \{3, -3, 5\}$ and $b = \{1, 2, 4\}$, what are the coordinates and magnitude of the vector c resulting from the cross product $a \times b = c$?

Answer: $c = a \times b = \{((-3)*4) - (5*2), ((5*1) - (3*4)), ((3*2) - ((-3)*1))\}$

$$c = \{-22, -7, 9\}$$

$$\|c\| = \sqrt{(-22)^2 + (-7)^2 + 9^2} \approx 24.78$$

A vector can also be multiplied with a square 3x3 matrix to produce another vector; however, this is a topic for another time.

Other Resources

A vast storehouse of reliable information about mathematics is available on the web from Wolfram, the makers of *Mathematica*, via <http://mathworld.wolfram.com>.

An index of Khan Academy videos related to vectors, vector operations, linear algebra and matrices is accessible via <http://www.khanacademy.org/math/linear-algebra/vectors>

<http://www.khanacademy.org/math/linear-algebra/v/vector-dot-product-and-vector-length>

<http://www.khanacademy.org/science/physics/v/the-dot-product>

<http://www.khanacademy.org/math/linear-algebra/v/linear-algebra--cross-product-introduction>

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<http://www.khanacademy.org/science/physics/v/dot-vs--cross-product>

<http://www.khanacademy.org/science/physics/v/calculating-dot-and-cross-products-with-unit-vector-notation>

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If you have any suggestions for improvement of this document, or notice any errors, please contact Vince Cronin via email at Vince_Cronin@baylor.edu.