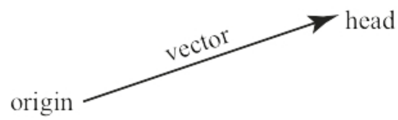
Vectors and Vector Arithmetic

Before we start... Khan Academy videos related to vectors and vector operations are accessible via http://www.khanacademy.org/math/linear-algebra/vectors?k Other reliable information about mathematics is available on the web from Wolfram, the makers of *Mathematica*, via http://mathworld.wolfram.com.

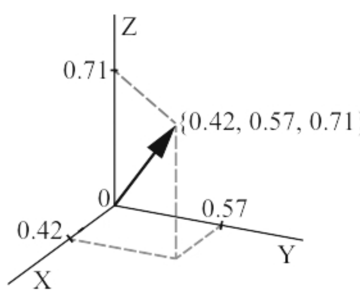
Definitions. A simple definition of a *vector* is that it is a mathematical entity that has a specific magnitude and direction. *Velocity* is a vector quantity, because it has *magnitude* (that is, displacement defined for a specified time interval) and *direction*. The magnitude of a velocity vector is called the *speed*, and is represented by a single number. Hence, speed is a *scalar* quantity, and is not a vector quantity because speed contains no information about the direction of movement.

Scalars and vectors are both members of the mathematical family of entities called *tensors*. A scalar is a 0th rank tensor, and a vector is a 1st rank tensor. Tuck these factoids away for future use when we define and start working with tensors.

Vectors are commonly visualized as directed line segments or arrows. The length of the arrow is the magnitude, and the direction that the arrow is pointing is the vector's direction. The vector's head is the pointed end of the arrow, and the vector's origin is the other end that corresponds to where the nock and feathers are on the weapon sort of arrow.



In order to know the direction that a vector is pointing, you need to specify a frame of reference. The direction of a vector on a horizontal plane might be specified by the azimuth or compass direction in which it is pointing. But many times, we will need to specify the direction of a 3-dimensional vector, so we need to refer to a 3D reference frame. Several such reference frames can be used, but we will generally use a Cartesian reference frame with three coordinate axes that are scaled identically and that are at right angles to each other.



If we take any vector and imagine its origin located at the origin of a 3D Cartesian coordinate system, everything we need to know about the vector can be derived from the coordinates of the point at the head of the vector. In the illustration above in which the vector is represented by an arrow, the X coordinate of the head of the vector is 0.42, the Y coordinate is 0.57, and the Z coordinate is 0.71. In other words, the coordinates of the head of the vector are {0.42, 0.57, 0.71}. If we call that vector *a*, then

$$a = \{0.42, 0.57. 0.71\}.$$

We say that "the coordinates of vector a are {0.42, 0.57. 0.71}," meaning that the origin of vector a is located at {0, 0, 0} and the head of vector a is located at {0.42, 0.57. 0.71}. We could move vector a so that it's origin was in a different location, say $\{x, y, z\}$, and then its head would be located at $\{0.42+x, 0.57+y, 0.71+z\}$ for any values of x, y or z, respectively. After any such translation, vector a would still have the same direction and magnitude.

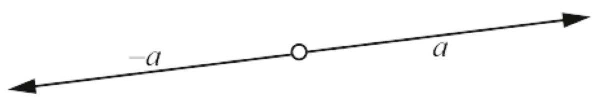
During a particular analysis, we might want to use symbols alone rather than inserting values for the vector coordinates. Two common ways of representing the vector coordinates symbolically are as follows:

$$a = \{a_x, a_y, a_z\},$$

which links each coordinate explicitly to coordinate axes that are labeled with the familiar X-Y-Z notation, or

$$a = \{a_1, a_2, a_3\}.$$

For a vector $a = \{a_1, a_2, a_3\}$, the opposite or negative of vector a (that is, $-a$) is a vector that has the same magnitude as a but is pointed in the opposite direction from a. The resulting coordinates for the negative of vector a are $-a = \{-a_1, -a_2, -a_3\}$



The length of a vector $a = \{a_1, a_2, a_3\}$ is signified by $|a|$

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

A *unit vector* is a vector whose magnitude is exactly 1. The unit vector of a vector a is signified by $\hat{a}$, and has coordinates

$$\hat{a} = \left\{ \frac{a_1}{|a|}, \frac{a_2}{|a|}, \frac{a_3}{|a|} \right\}$$

Vector Arithmetic

In the following subsections, we will use the following vectors:

$$a = \{a_1, a_2, a_3\}$$

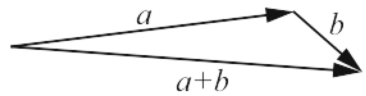
$$b = \{b_1, b_2, b_3\}$$

$$c = \{c_1, c_2, c_3\}$$

Vector addition. In vector addition, you add the first component of the two vectors together to find the first component of the vector sum, then do the same for the second and third components.

$$a + b = \{(a_1 + b_1), (a_2 + b_2), (a_3 + b_3)\}$$

Graphically, vector addition looks like this:



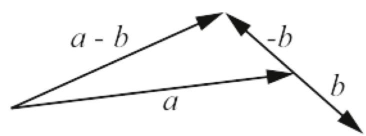
If you have a set of *n* vectors (where *n* is an integer that is greater than 1), the result of adding all *n* vectors together is a vector whose direction is the mean (average) direction of that set of vectors. For example, the mean vector of the set of $n = 3$ vectors that includes vectors *a*, *b* and *c* is given by

$$a + b + c = \{(a_1 + b_1 + c_1), (a_2 + b_2 + c_2), (a_3 + b_3 + c_3)\}$$

Vector subtraction. In vector subtraction, you subtract the first component of the first vector from the first component of the second vector to find the first component of the vector difference, then do the same for the second and third components.

$$a - b = \{(a_1 - b_1), (a_2 - b_2), (a_3 - b_3)\}$$

This is equivalent to adding vector *a* to the opposite of vector *b* (that is, adding *a* to –*b*). Graphically, vector subtraction looks like this:



Vector multiplication. We will not consider the details of vector multiplication in this handout, except to say that vectors can be "multiplied" in several senses. A vector *dot product* yields a single number, and can be used to determine the angle between two vectors. Because the dot product yields a single number, it is also called a scalar product. The dot product of two different vectors *a* and *b* is usually represented by the symbol $a \cdot b$.

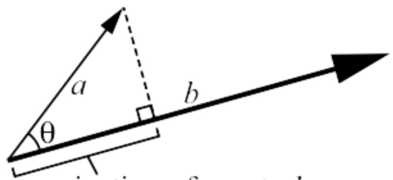
$$a \cdot b = (a_1 b_1) + (a_2 b_2) + (a_3 b_3), \text{ and}$$

$$a \cdot b = |a||b|\cos\theta,$$

where $|a|$ is the length of vector *a*, $|b|$ is the length of vector *b*, and θ is the angle between vectors *a* and *b*. The result of a dot product is a scalar, because all three of the entities that are multiplied together ($|a|$, $|b|$, and $\cos\theta$) are scalars. The product of scalars is a scalar. The angle θ is given by

$$\theta = \cos^{-1}\left[\frac{a \cdot b}{|a||b|}\right]$$

A good video about the dot product is available from the Khan Academy at http://www.khanacademy.org/science/physics/v/the-dot-product. That video emphasizes that a dot product $a \cdot b$ is the product of the projection of vector *a* onto *b* ($|a|\cos\theta$) times the length of b ($|b|$), which is also equal to the projection of *b* onto *a* ($|b|\cos\theta$) times the length of *a* ($|a|$). "The projection of vector *a* onto *b*" might also be described as the component of vector *a* that is parallel to vector *b*.



projection of a onto b, or component of a parallel to b

The dot product $a \cdot b$ is equal to the length of vector b times the length of the projection of vector a onto vector b

The vector cross product, $a \times b$, results in a third vector whose direction is perpendicular to the plane defined by vectors *a* and *b*.

$$a \times b = \{(a_2 b_3) - (a_3 b_2), (a_3 b_1) - (a_1 b_3), (a_1 b_2) - (a_2 b_1)\}\text{, and}$$

$$(a \times b) = -(b \times a)$$

The magnitude is

$$|a \times b| = |a||b| \sin\theta$$

where $|a|$ is the length of vector *a*, $|b|$ is the length of vector *b*, and θ is the angle between vectors *a* and *b*.

A vector can also be multiplied with a square 3x3 matrix to produce another vector; however, this is a topic for another day.