Eigen decomposition for 3D stress tensor

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Unfinished Draft; Subject to Revision

Introduction

If we have a cubic free body that is arbitrarily oriented in a stress field and is in a state of static equilibrium, and we know the tractions (normal and shear stresses) acting on each of its infinitesimal faces, we can use eigen decomposition of the traction tensor to define the orientation and magnitude of the corresponding principal stresses. The orientation of the orthogonal basis vectors that are parallel to the principal axes are the eigenvectors of the symmetric traction tensor, and the magnitudes of the principal stresses (σ_1, σ_2, σ_3) are the corresponding eigenvalues.

Imagine that we have a symmetrical $3-\mathrm{D}$ matrix representing the traction tensor as follows :

$$\mathbf{s} = \begin{pmatrix} 10 & 3 & 6 \\ 3 & 4 & 7 \\ 6 & 7 & 2 \end{pmatrix}$$

We will use the *Mathematica* function **Eigenvectors** to define the eigenvectors of matrix **s**:

```
Eigenvectors[s];
```

Similarly, we can use the *Mathematica* function **Eigenvalues** to define the eigenvectors of matrix **s**

```
Eigenvalues[s];
```

Instructions

This code requires six input data : the stress acting normal to the X plane (σ_{xx}), the stress acting normal to the Y plane (σ_{yy}), the stress acting normal to the Z plane (σ_{zz}), the shear stress acting on the X plane in the Y direction (σ_{xy}), the shear stress acting on the X plane in the Z direction (σ_{xz}), and the shear stress acting on the Y plane in the Z direction (σ_{yz}). In accord with the convention generally

used by geologists, a compressive normal stress has a positive sign, tensional normal stress is negative, and a shear stress that tends to make a body rotate in an anticlockwise direction is a positive shear stress.
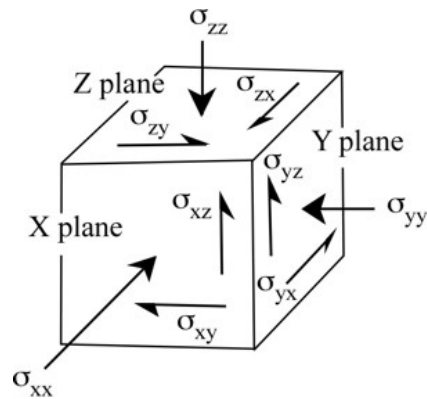
Input

 In the input section below, type the stress values on the appropriate lines of code (which are shaded light red) after the "=" sign and before the ";" mark.

The variable σ_{xx} is the normal stress acting on the X plane -- the plane that is perpendicular to the X coordinate axis of the free body diagram. By the structural geologists' sign convention, a compressive normal stress is a positive value, and a tensional normal stress is a negative value.

```
σxx = 2;
```

The variable σ_{yy} is the normal stress acting on the Y plane -- the plane that is perpendicular to the Y coordinate axis of the free body diagram. By the structural geologists' sign convention, a compressive normal stress is a positive value, and a tensional normal stress is a negative value.

```
σyy = 1.5;
```

The variable σ_{zz} is the normal stress acting on the Z plane -- the plane that is perpendicular to the Z coordinate axis of the free body diagram. By the structural geologists' sign convention, a compressive normal stress is a positive value, and a tensional normal stress is a negative value.

```
σzz = 1.7;
```

The variable σ_{xy} is the shear stress acting on the X plane in the Y direction. By the structural geologists' sign convention, a shear stress has a positive value if it tends to rotate the body in an anticlockwise direction around the origin of the coordinate system.

```
σxy = 0.7;
```

The variable σ_{xy} is the shear stress acting on the X plane in the Y direction. By the structural geologists' sign convention, a shear stress has a positive value if it tends to rotate the body in an anticlockwise direction around the origin of the coordinate system.

```
σxz = 0.3;
```

The variable σ_{xy} is the shear stress acting on the X plane in the Y direction. By the structural geologists' sign convention, a shear stress has a positive value if it tends to rotate the body in an anticlockwise direction around the origin of the coordinate system.

```
σyz = 0.5;
```

Computation

```
σyx = σxy;
```

```
σzx = σxz;
```

```
σzy = σyz;
```

$$s = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix};$$

The eigenvectors that result from the next step are unit vectors that are colinear with the principal axes of the ellipse. Consequently, the three eigenvectors ({result1[[1, 1]], result1[[1, 2]], result1[[1, 3]]}, {result1[[2, 1]], result1[[2, 2]], result1[[2, 3]]} and {result1[[3, 1]], result1[[3, 2]], result1[[3, 3]]}) are perpendicular to each other. The eigenvalues are the magnitudes of the principal stresses.

```
principalAxes = Eigenvectors[s];
```

```
axisLengths = Sort[Eigenvalues[s], Greater];
```

The variable "a" is the length of the major axis of the principal ellipse; "b" is the length of the intermediate axis; "c" is the length of the minor axis.

```
σ1 = axisLengths[[1]];
```

```
σ2 = axisLengths[[2]];
```

```
σ3 = axisLengths[[3]];
```

Results

The eigenvectors are

```
principalAxes
```

```
{{-0.690205, -0.561248, -0.456746},
 {-0.579638, 0.0509703, 0.813279}, {0.433171, -0.826076, 0.360501}}
```

The magnitude of the greatest principal stress (σ_1) in megaPascals is

σ_1

```
2.76774
```

The corresponding axial vector is

**vectorσ_1 = $\{$(σ_1 principalAxes[[1, 1]]),
  (σ_1 principalAxes[[1, 2]]), (σ_1 principalAxes[[1, 3]])$\}$**

```
{-1.91031, -1.55339, -1.26415}
```

The magnitude of the intermediate principal stress (σ_2) in megaPascals is

σ_2

```
1.51752
```

The corresponding axial vector is

**vectorσ_2 = $\{$(σ_2 principalAxes[[2, 1]]),
  (σ_2 principalAxes[[2, 2]]), (σ_2 principalAxes[[2, 3]])$\}$**

```
{-0.879612, 0.0773485, 1.23417}
```

The magnitude of the least principal stress (σ_3) in megaPascals is

σ_3

```
0.914739
```

The corresponding axial vector is

**vectorσ_3 = $\{$(σ_3 principalAxes[[3, 1]]),
  (σ_3 principalAxes[[3, 2]]), (σ_3 principalAxes[[3, 3]])$\}$**

```
{0.396239, -0.755645, 0.329764}
```

Graphics

The parametric equations of an ellipsoid can be written as
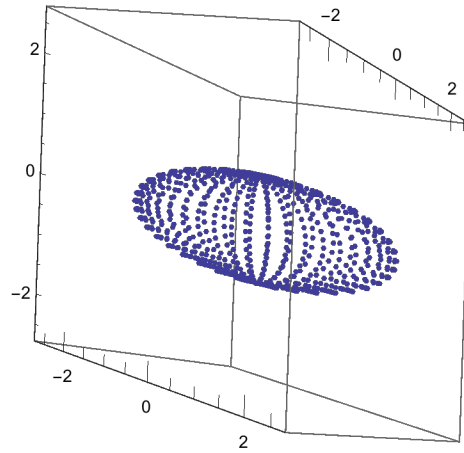$x = a$ Cos[u] Sin[v]
$y = b$ Sin[u] Sin[v]
$z = c$ Cos[v]
for u in[0, 2 π] and v in[0, π].

```
test1 = Table[{σ₁ Cos[u] Sin[v], σ₂ Sin[u] Sin[v], σ₃ Cos[v]},
    {u, 0, 2π, (π/20)}, {v, 0, π, (π/20)}];
```

```
test2 = Flatten[test1, 1];
```

```
ListPointPlot3D[test2, AspectRatio → 1,
  PlotRange → {{-σ₁, σ₁}, {-σ₁, σ₁}, {-σ₁, σ₁}}]
```

```
test4 = ParametricPlot3D[{σ₁ Cos[u] Sin[v], σ₂ Sin[u] Sin[v], σ₃ Cos[v]},
  {u, 0, 2 π}, {v, 0, π}, ColorFunction → "GrayTones",
  PlotStyle → Specularity[White, 50], AxesLabel → {"Greatest Principal Stress",
    "Intermediate Principal Stress", "Least Principal Stress"}, Mesh → None,
  AspectRatio → 1, PlotRange → {{-σ₁, σ₁}, {-σ₁, σ₁}, {-σ₁, σ₁}}]
```
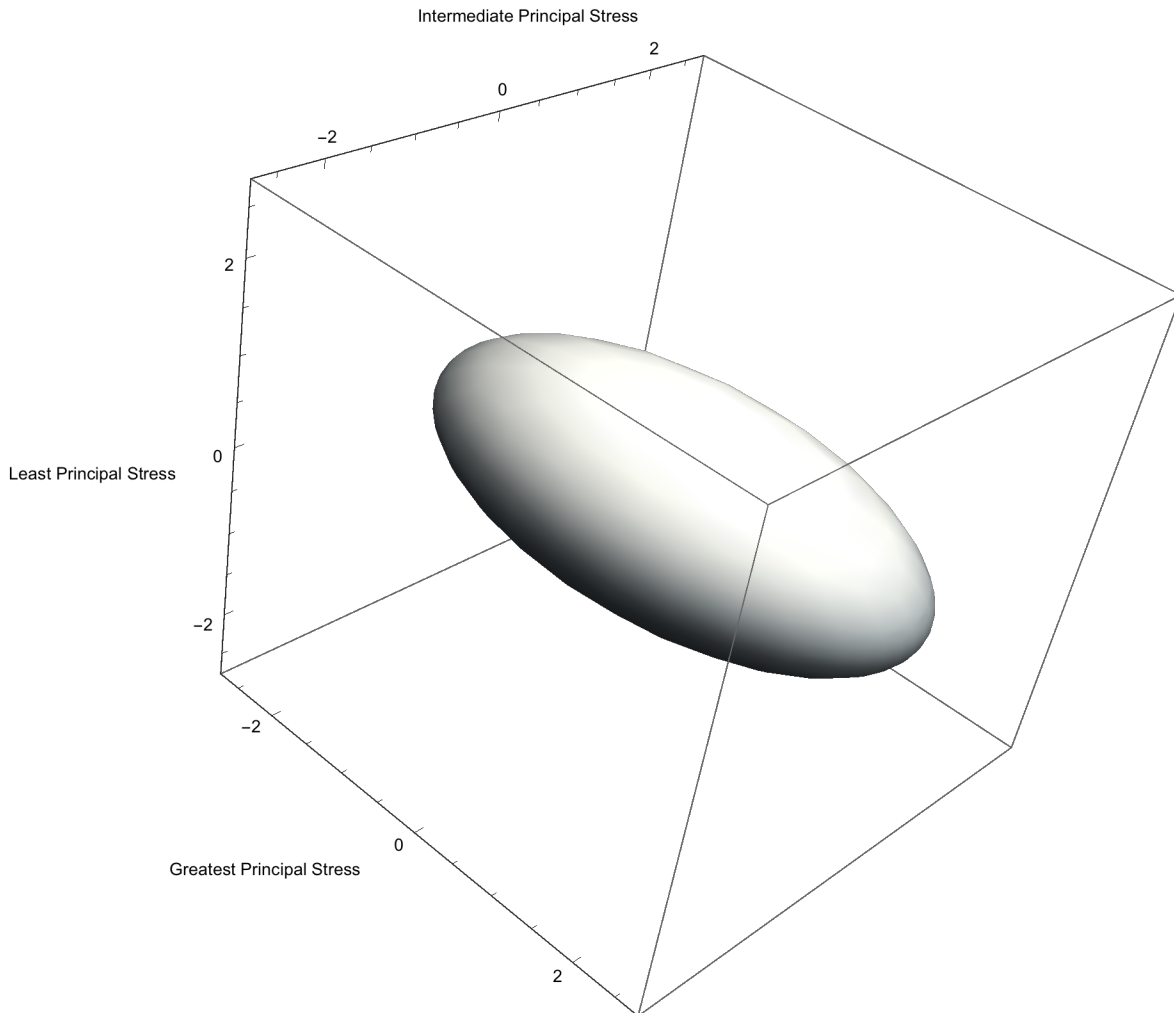

References

Ferguson, John, 1994, **Introduction to Linear Algebra in Geology** : London, Chapman & Hall, 203 p., ISBN 0 - 412 - 49350 - 0

Fossen, Haakon, 2010, **Structural Geology** : New York, Cambridge University Press, 463 p., ISBN 978 - 0 - 521 - 51664 - 8

Oertel, G., 1996, **Stress and Deformation -- A Handbook on Tensors in Geology** : New York, Oxford

University Press, 292 p., ISBN 0 - 19 - 509503 - 0

Pollard, D.D., and Fletcher, R.C., 2005, **Fundamentals of Structural Geology** : New York, Cambridge University Press, 500 p., ISBN 0 - 521 - 83927 - 0; see especially Chapter 6.

The graphics code to produce the ellipsoid came from Eric W.Weisstein, 2009,
http : // mathworld.wolfram.com / notebooks / Surfaces / Ellipsoid.nb.
For more information,
see Eric' s *MathWorld* entry http : // mathworld.wolfram.com / Ellipsoid.html.

Wolfram MathWorld has a web resource on eigenvectors and eigen decomposition available at
http : // mathworld.wolfram.com/Eigenvector.html

The corresponding resource on ellipsoids is available at
http://mathworld.wolfram.com/Ellipsoid.html

Color schemes used in *Mathematica* are displayed at
http://reference.wolfram.com/mathematica/guide/ColorSchemes.html

Other *Mathematica* resources are available for download at
Emmanuel Amiot, "Eigenvectors by Hand" from The Wolfram Demonstrations Project http://demonstrations.wolfram.com/EigenvectorsByHand/
Yaroslav Bulatov, "Linear Transformation with Given Eigenvectors" from The Wolfram Demonstrations Project http://demonstrations.wolfram.com/LinearTransformationWithGivenEigenvectors/