Matrices and Matrix Math

It will be useful to remember (or perhaps learn) a few things about matrices so we can use them to solve problems. We will start by defining a matrix we will call *a*.

$$a = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

Matrix *a* is called a *column matrix*, because its three elements are arrayed in a vertical column. Matrix *a* might also be called a 3x1 matrix, because its elements are arrayed in three rows and 1 column. The subscripts associated with each of the elements in the array indicate the position of the element in the array, so a_{21} is the element in the 2nd row and 1st column. The first subscript indicates what row the element is located in, and the second subscript indicates the column. The mnemonic is "rc," as in "Roman Catholic," but really it stands for "row column." We often associate the subscript "1" with an *x* coordinate axis, "2" with a *y* axis, and "3" with a *z* axis.

A matrix with one row and three columns (a *row matrix* or a 1x3 matrix) would look like this

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \end{bmatrix},$$

and a matrix with three rows and two columns (a 3x2 matrix) would look like this

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix}.$$

Matrix *b* is a 3x3 matrix because it has three rows and three columns. It is also known as a *square matrix*, because it has the same number of rows as columns.

$$b = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Some simple matrix mathematics

We can add, subtract, multiply or divide each of the components of a matrix by a scalar. For example, if *s* is a scalar and *b* is a 2x2 matrix,

$$s + b = s + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} s + b_{11} & s + b_{12} \\ s + b_{21} & s + b_{22} \end{bmatrix}$$

We can also add, subtract, multiply or divide two matrices that have the same dimensions, component-by-component. For example, if *b* and *c* are both 2x2 matrices,

$$b \div c = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \div \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} b_{11}/c_{11} & b_{12}/c_{12} \\ b_{21}/c_{21} & b_{22}/c_{22} \end{bmatrix}$$

Multiplication of two matrices b and c by components is only possible if the two matrices have the same number of rows and columns (that is, they have the same *dimensions*), and might be indicated by $b*c$

$$b * c = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} * \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} b_{11}*c_{11} & b_{12}*c_{12} \\ b_{21}*c_{21} & b_{22}*c_{22} \end{bmatrix}$$

Probably the more commonly used mode of matrix multiplication, distinct from simple multiplication by component as described above, involves the use of dot products. Two matrices can be multiplied using dot products if the number of rows in one matrix equals the number of columns in the other matrix. The matrices to be multiplied in this manner do not need to have the same dimensions or number of components; however, one matrix must have the same number of rows as the other matrix has columns. Let's multiply matrices a and b together using dot products to yield a product: matrix d.

$$c = b \cdot a$$

or, represented another way,

$$\begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}.$$

We can think of matrix b as consisting of three 3-component vectors: $\{b_{11}, b_{12}, b_{13}\}$ in the top row, $\{b_{21}, b_{22}, b_{23}\}$ in the middle row, and $\{b_{31}, b_{32}, b_{33}\}$ in the bottom row. Each element of the resultant matrix d is the dot product of matrix a with one row-vector of matrix b.

$$\begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \end{bmatrix} = \begin{bmatrix} topRow \cdot a \\ middleRow \cdot a \\ bottomRow \cdot a \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \end{bmatrix} = \begin{bmatrix} (b_{11}a_{11})+(b_{12}a_{21})+(b_{13}a_{31}) \\ (b_{21}a_{11})+(b_{22}a_{21})+(b_{23}a_{31}) \\ (b_{31}a_{11})+(b_{32}a_{21})+(b_{33}a_{31}) \end{bmatrix}.$$

For example, the top element in matrix d is found by solving the following equation

$$d_{11} = (b_{11} \times a_{11}) + (b_{12} \times a_{21}) + (b_{13} \times a_{31}).$$

And so the product (matrix d) of multiplying a 3x1 vector (matrix a) by a 3x3 matrix (matrix b) is a matrix with three elements: another vector. What if we take vectors a, b and e and multiply them together to yield a matrix f, where a and b are the same as before and matrix e is

$$e = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}.$$

It turns out that the equation

$$f = e \cdot b \cdot a$$

is functionally the same as

$$f = e \cdot d$$

where

$$d = b \cdot a$$

as we defined matrix d above, so

$$\begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \end{bmatrix} = \begin{bmatrix} eTopRow \cdot d \\ eMiddleRow \cdot d \\ eBottomRow \cdot d \end{bmatrix} = \begin{bmatrix} (e_{11}d_{11}) + (e_{12}d_{21}) + (e_{13}d_{31}) \\ (e_{21}d_{11}) + (e_{22}d_{21}) + (e_{23}d_{31}) \\ (e_{31}d_{11}) + (e_{32}d_{21}) + (e_{33}d_{31}) \end{bmatrix}.$$

In these examples, we start the process of multiplying from the right side of the equation, where we will find a 3x1 matrix representing a vector. Multiplying a 3x1 matrix by the next matrix to the left (a 3x3 matrix) yields another 3-component matrix representing a vector. If there is another 3x3 matrix to the left, repeat the process, and keep repeating the process until you reach the = sign.

Example. Find the result of the following matrix multiplication:

$$\begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 8 \\ 9 & 7 & 5 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.5 \\ 0.8 & 0.9 & 0.7 \\ 0.4 & 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 11 \\ 15 \\ 18 \end{bmatrix}$$

Solution, step 1. Start by multiplying the 3x1 vector matrix on the right by the 3x3 matrix next to it, in the middle of the sequence.

$$\begin{bmatrix} 20.2 \\ 34.9 \\ 11.3 \end{bmatrix} = \begin{bmatrix} (0.2 \times 11) + (0.6 \times 15) + (0.5 \times 18) \\ (0.8 \times 11) + (0.9 \times 15) + (0.7 \times 18) \\ (0.4 \times 11) + (0.1 \times 15) + (0.3 \times 18) \end{bmatrix} = \begin{bmatrix} 0.2 & 0.6 & 0.5 \\ 0.8 & 0.9 & 0.7 \\ 0.4 & 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 11 \\ 15 \\ 18 \end{bmatrix}$$

Step 2. Use the results of step 1 as the 3x1 vector matrix on the right.

$$\begin{bmatrix} 268 \\ 180.4 \\ 482.6 \end{bmatrix} = \begin{bmatrix} (3 \times 20.2) + (4 \times 34.9) + (6 \times 11.3) \\ (1 \times 20.2) + (2 \times 34.9) + (8 \times 11.3) \\ (9 \times 20.2) + (7 \times 34.9) + (5 \times 11.3) \end{bmatrix} = \begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 8 \\ 9 & 7 & 5 \end{bmatrix} \begin{bmatrix} 20.2 \\ 34.9 \\ 11.3 \end{bmatrix}$$

The product of the three matrices is the following 3-component vector: {268, 180.4, 482.6}.

Recognizing different parts of a matrix and different types of matrix

In this square matrix,

$$\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

the part of the matrix that has all of the capital letters (*A, B, C*) is called the *diagonal* or *axis* of the matrix. Values that are in the positions occupied by the 0s are said to be *off-axis* terms.

In a *symmetric* matrix, like the one below, the values above the diagonal are equal to the values below and directly across the diagonal.

$$\begin{bmatrix} A & d & e \\ d & B & f \\ e & f & C \end{bmatrix}$$

In an *antisymmetric* matrix, the values across the diagonal from each other have the same magnitude but different sign.

$$\begin{bmatrix} A & d & e \\ -d & B & f \\ -e & -f & C \end{bmatrix}$$

An *asymmetric* matrix, like

$$\begin{bmatrix} A & d & e \\ g & B & h \\ n & -f & C \end{bmatrix}$$

lacks at least some of the symmetries we have just examined.

If we define a matrix *M* as follows

$$M = \begin{bmatrix} A & d & e \\ g & B & h \\ n & -f & C \end{bmatrix},$$

the *transpose* of matrix *M* is represented by M^T and is

$$M^T = \begin{bmatrix} A & g & n \\ d & B & -f \\ e & h & C \end{bmatrix}.$$

The values along the diagonal of the transposed matrix are unchanged from the original matrix, but the values across the diagonal from each other are swapped.

The *inverse* of a matrix M is symbolized by M^{-1}. If a matrix is multiplied by its inverse, the result is the identity matrix whose diagonal terms are all 1s and whose off-axis terms are all 0s.

$$M \cdot M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If the transpose of a matrix is the same as the inverse of the matrix, that matrix is called an *orthogonal* matrix.

Resources

A summary sheet about vectors that includes information about dot products is available from the same source from which you obtained this summary sheet.

Davis, H.F., and Snider, A.D., 1987, Introduction to vector analysis [fifth edition]: Boston, Allyn and Bacon, 365 p. ISBN 0-205-10263-8.

Web resources

There are several videos available online from the Khan Academy (http://www.khanacademy.org) that relate to various aspects of matrices, matrix mathematics, vectors and linear algebra. For example, an introduction to matrices is available at http://www.khanacademy.org/math/algebra/algebra-matrices/v/introduction-to-matrices

Weisstein, Eric W., Matrix: *MathWorld*--A Wolfram Web Resource, accessed 2 September 2012 via http://mathworld.wolfram.com/Matrix.html

Weisstein, Eric W., Matrix inversion: *MathWorld*--A Wolfram Web Resource, accessed 2 September 2012 via http://mathworld.wolfram.com/MatrixInversion.html

Weisstein, Eric W., Matrix multiplication: *MathWorld*--A Wolfram Web Resource, accessed 2 September 2012 via http://mathworld.wolfram.com/MatrixMultiplication.html

Weisstein, Eric W., Vector: *MathWorld*--A Wolfram Web Resource, accessed 2 September 2012 via http://mathworld.wolfram.com/Vector.html

Weisstein, Eric W., Vector multiplication: *MathWorld*--A Wolfram Web Resource, accessed 2 September 2012 via http://mathworld.wolfram.com/VectorMultiplication.html